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Determination of the metric tensor from components of the Riemann tensor

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Abstract. An algebraic method is presented which shows how to determine the components of the metric tensor $g_{\mu\nu}$, up to an arbitrary conformal factor, from a given set of components of its Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$ in some coordinate frame. This procedure follows and generalises one given by Ihrig. Since the computations are purely algebraic and are carried out at a point in the manifold, no differentiability or continuity conditions are assumed. A number of examples are given to illustrate the technique. Although in general the method determines the metric up to one arbitrary scalar, the conformal factor, in a number of cases either one or three other arbitrary scalars arise. However, these latter cases are rare, and the form of the Riemann tensor for such cases have been listed elsewhere.

1. Introduction

An algebraic method for constructing the components of the metric tensor $g_{\mu\nu}$, up to a conformal factor, from the components of its corresponding Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}$ in some coordinate frame from the identity

$$g_{\mu(\nu}R^{\mu}{}_{\lambda)\alpha\beta}=0 \tag{1.1}$$

has been given by Ihrig (1975*a*). The procedure works at a point in an n-dimensional Riemannian or pseudo-Riemannian manifold only in those cases where, in Ihrig's language, the Riemann tensor is *total* at that point. The emphasis in this paper is on space–times, so that the theory discussed and the examples given are for four-dimensional pseudo-Riemannian manifolds. However, the comments here could be extended in obvious ways to the more general cases.

The method also works for any fourth-order tensor of type (1, 3) which possesses the skew symmetry represented by the identity (1.1). Thus it works for a given set of components of the Weyl tensor, and some theorems on the Weyl tensor similar to theorems on the Riemann tensor given in related papers can be easily written down. Generalisations can also be made to tensors of other orders which satisfy equations similar to (1.1).

The curvature 2-forms $\Omega^{a}{}_{b}$ of a space–time are defined in terms of the tetrad components $R^{a}{}_{bcd}$ of the Riemann tensor by

$$\Omega^{a}{}_{b}=\tfrac{1}{2}R^{a}{}_{bcd}\theta^{c}\wedge\theta^{d} \tag{1.2}$$

where the θ^{a} are basis 1-forms. The requirement that the Riemann tensor be *total* is

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equivalent to the requirement that the bivector space which spans the curvature 2-forms has dimension six. This requirement was shown by McIntosh and Halford (1981) to be unnecessarily strong in that the algebraic equation

$$x_{\mu(\nu}R^{\mu}{}_{\lambda)\alpha\beta}=0, \qquad x_{[\mu\nu]}=0 \tag{1.3}$$

admits only the trivial solution

$$x_{\mu\nu}=\phi g_{\mu\nu}, \tag{1.4}$$

where ϕ is an arbitrary scalar, whenever the dimension of the bivector space is four or more. This is equivalent to saying that, at a point, the only algebraic equation of the form (1.3) satisfied by the $R^{\mu}{}_{\nu\alpha\beta}$ in these cases is the identity (1.1), which merely says that $R_{\mu\nu\alpha\beta}=-R_{\nu\mu\alpha\beta}$. But, as discussed by Hall and McIntosh (1981), even this requirement is too strong—it can be weakened to the following: The equation (1.3) admits non-trivial solutions (i.e. not of the form (1.4)) when either (*a*) there is at least one (null or non-null) vector $\boldsymbol{V}$ which satisfies

$$V_{\mu}R^{\mu}{}_{\nu\alpha\beta}=0, \tag{1.5}$$

or (*b*) the curvature 2-forms are spanned by two bivectors

$$\theta^{0}\wedge\theta^{1}, \qquad \theta^{2}\wedge\theta^{3}, \tag{1.6}$$

where the θ^{a} are orthogonal vectors with θ^{0} timelike and the $\theta^{i}(i=1,2,3)$ spacelike. The dimension of the bivector space which spans the curvature 2-forms is three if there is one vector which satisfies equation (1.5), or one if there are two such linearly independent vectors. There cannot be three such independent vectors satisfying (1.5) unless the space–time is flat.

It is to be emphasised that the method under discussion for determining the $\phi g_{\mu\nu}$ from the $R^{\mu}{}_{\nu\alpha\beta}$ is used *at a point.* Hall and McIntosh (1981) have remarked upon this to the effect that the holonomy group of the manifold may not necessarily play a role in the process. However, the holonomy group has been introduced into the discussion of equations (1.3) and (1.4) by Hlavatý (1959a, b, 1960) and Ihrig (1975a, b, 1976), who assumed appropriate differentiability conditions and commented on physically relevant cases.

The derivation of the $\phi g_{\mu\nu}$ from a given set of $R^{\mu}{}_{\nu\alpha\beta}$ by using Ihrig's construction appears to be very involved. In practice it is very simple and can be extended easily to the exceptional cases in which the $\phi g_{\mu\nu}$ are given by (1.1). The first four examples in the next section show how the construction and its extension can be used to generate space–time metrics from Riemann tensor components. The metrics concerned are:

(*a*) the Schwarzschild metric, in which case $\phi g_{\mu\nu}$ is given explicitly from the $R^{\mu}{}_{\nu\alpha\beta}$;

(*b*) the vacuum Petrov type N *pp*-wave metric (2.9), when (1.1) gives $\phi g_{\mu\nu}+\alpha l_{\mu}l_{\nu}$, where $\boldsymbol{l}$ is a repeated principal null vector of the Weyl tensor and α is a second arbitrary scalar;

(*c*) the vacuum Petrov type N plane-fronted wave metric with rotation, (2.21) with $\phi=1$, in which case $\phi g_{\mu\nu}$ is again not given by (1.1). However, in this case if the covariant derivative components $R^{\mu}{}_{\nu\alpha\beta;\gamma}$ are known, then the identity

$$g_{\mu(\nu}R^{\mu}{}_{\lambda)\alpha\beta;\gamma}=0 \tag{1.7}$$

can be used together with (1.1) to give $g_{\mu\nu}$ up to a conformal factor;

(*d*) a metric whose curvature 2-form is spanned as in (1.6).

The difference between (*b*) and (*c*) is that, with differentiability assumed, the dimension of the bivector space of the Riemann tensor and its derivatives in (*c*) is six, while in (*b*) it is only two. This is apparent from the work of Goldberg and Kerr (1961), who, in discussing the role of the holonomy group, showed that the group dimension is two in case (*b*) and six in case (*c*). However, in case (*c*) the holonomy group is imperfect because the dimension of the bivector space of the Riemann tensor alone is less than six—indeed it is only two.

A fifth example shows an attempt to use Ihrig's method when the Riemann tensor components are not those of a metric. The case considered is that of Newtonian gravity for which there is no four-dimensional metric.

2. Method and examples

Before detailing the examples, we outline the steps involved in Ihrig's method and our extension of it.

Method. Consider a vector space X spanned by ten orthonormal vectors $x_{\mu\nu} = x_{\nu\mu}$ with the inner product

$$(x_{\mu_1\nu_1}, x_{\mu_2\nu_2}) = \delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}. \tag{2.1}$$

Greek indices range over 0, 1, 2, 3.

Step 1: In X form as many linearly independent vectors $\boldsymbol{v}_a (a = 1, 2, \ldots, m; m \leq 9)$ as possible from

$$\psi x_{\mu(\nu} R^{\mu}{}_{\lambda)\alpha\beta}. \tag{2.2}$$

Here ψ is an arbitrary scalar function. The lengths of these v_a are not important, and so ψ is chosen to give the $\boldsymbol{v}_a$ simple forms as linear combinations of the $x_{\mu\nu}$.

Step 2: Write down the most general vector $\boldsymbol{\omega}$ in X which is orthogonal to the vectors $\boldsymbol{v}_a$. In practice $\boldsymbol{\omega}$ can have one, two or three arbitrary functions in it, although usually it has only one.

Step 3: Calculate the metric components from the inner product

$$\lambda g_{\mu\nu} = (\boldsymbol{\omega}, x_{\mu\nu}), \tag{2.3}$$

where λ is an arbitrary scalar.

Example 1. The Schwarzschild metric. Given the Riemann tensor components

$$\begin{aligned}
&R^{0}{}_{220} = R^{1}{}_{221} = \tfrac{1}{2}R^{3}{}_{232} = m\rho,\\
&R^{0}{}_{330} = R^{1}{}_{331} = \tfrac{1}{2}R^{2}{}_{323} = m\rho \sin^2\theta,\\
&R^{2}{}_{112} = R^{3}{}_{113} = \tfrac{1}{2}R^{0}{}_{101} = m\rho^3 A^{-1},\\
&R^{2}{}_{020} = R^{3}{}_{030} = \tfrac{1}{2}R^{1}{}_{001} = m\rho^3 A,\\
&R^{\mu}{}_{\nu\beta\alpha} = -R^{\mu}{}_{\nu\alpha\beta},
\end{aligned} \tag{2.4}$$

where

$$x^{\mu} = (t, r, \theta, \varphi), \qquad \rho = 1/r, \qquad A = 1 - 2m\rho, \tag{2.5}$$

it is now a matter of looking at values of $\nu\lambda\alpha\beta$ in (2.2) to find as many linearly independent vectors v_a as possible. For instance, for $\nu\lambda\alpha\beta = 0101$ we find

$$v_1 = \psi(x_{00}R^0{}_{101} + x_{11}R^1{}_{001}).$$

Choose ψ so that, for example,

$$v_1 = A^{-1}x_{00} + Ax_{11}.$$

Similarly (2.2) yields

$$\begin{aligned} &v_2 = x_{01}, \qquad v_3 = x_{02}, \qquad v_4 = x_{03}, \\ &v_5 = x_{12}, \qquad v_6 = x_{13}, \qquad v_7 = x_{23}, \\ &v_8 = A^{-1}x_{00} + \rho^2 x_{22}, \qquad v_9 = A^{-1}\sin^2\theta x_{00} + \rho^2 x_{33}. \end{aligned} \tag{2.6}$$

Notice that is is useful to find a vector like $v_2 = x_{01}$ early in the calculation since the v_a can be chosen to be orthogonal and thus any x_{01} term can be omitted from all other vectors v_a. The construction ensures that, with one of the v_a proportional to a single term x_{01}, the final form of the metric will have $g_{01} = 0$. The vector in X which is orthogonal to all the v_a in (2.6) is

$$\omega = Ax_{00} - A^{-1}x_{11} - r^2(x_{22} + \sin^2\theta x_{33}). \tag{2.7}$$

Equation (2.3) now gives

$$ds^2 = \phi[-A\,dt^2 + A^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]. \tag{2.8}$$

The arbitrary scalar ϕ can now be found from the Riemann tensor components by differentiation, provided that the differentiability conditions can be satisfied. It can also be found from the Bianchi identities; see Ihrig (1975b).

Example 2: The vacuum *pp*-wave metric. This has the form

$$ds^2 = [F(u, \zeta) + \bar{F}(u, \bar{\zeta})]\,du^2 + 2\,du\,dv - 2\,d\zeta\,d\bar{\zeta}, \tag{2.9}$$

where F is an arbitrary complex function of its arguments. With coordinates $x^\mu = (u, v, \zeta, \bar{\zeta})$, the only non-zero components of the Riemann tensor are

$$R^1{}_{202} = -R^1{}_{220} = R^3{}_{002} = -R^3{}_{020} = \tfrac{1}{2}F_{,\zeta\zeta} \tag{2.10}$$

and their complex conjugates (obtained by interchanging indices 2 and 3). Suppose we are given the $R^\mu{}_{\nu\alpha\beta}$ as in (2.10). Then a calculation similar to that in example 1 gives

$$v_1 = x_{01} + x_{23}, \tag{2.11}$$

and v_2 to v_8 equal to x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{22} and x_{33}. Then the vector orthogonal to all of these v_a is

$$\omega = \alpha x_{00} + \beta(x_{01} - x_{23}), \tag{2.12}$$

and equation (2.3) gives

$$ds^2 = \phi(du\,dv - d\zeta\,d\bar{\zeta}) + \psi\,du^2, \tag{2.13}$$

where ϕ, ψ, α and β are arbitrary scalar functions. Another way of interpreting (2.13) is to say that, for this metric, the only possible solutions of (1.3) are

$$x_{\mu\nu} = \phi g_{\mu\nu} + \alpha l_\mu l_\nu, \tag{2.14}$$

where $l = \partial/\partial v$ is the repeated principal null vector of the Weyl tensor formed from (2.9). This result is due to Collinson (1970).

Example 3. The vacuum type N plane-fronted rotating metric. The only non-zero independent components of the Riemann tensor are

$$R^1{}_{220} = -\tfrac{1}{2}(U\xi^2 - 2U_{,\zeta}\xi + 9v^2\xi^4), \qquad R^1{}_{330} = \overline{R^1{}_{220}}, \qquad R^1{}_{020} = 2v\xi R^1{}_{220}, \qquad R^3{}_{020} = 2R^1{}_{220}, \qquad R^1{}_{030} = \overline{R^1{}_{020}}, \qquad R^2{}_{030} = 2R^1{}_{330} \tag{2.15}$$

together with $R^1{}_{202}$, etc. Here $x^\mu = (u, v, \zeta, \bar{\zeta})$ and

$$U(u, v, \zeta, \bar{\zeta}) = -3v^2\xi^2/2 + x[F(u, \zeta) + \bar{F}(u, \bar{\zeta})], \qquad \zeta = x + \mathrm{i}y, \qquad \xi = 1/x, \tag{2.16}$$

where F is an arbitrary complex function of u and ζ. As in the previous example, the actual form of $R^1{}_{220}$ is unimportant since it does not appear in the form of the $g_{\mu\nu}$. A calculation of the kind outlined before gives

$$\boldsymbol{v}_1 = v\xi x_{01} + x_{03}, \qquad \boldsymbol{v}_2 = x_{01} + 2x_{23}, \qquad \boldsymbol{v}_3 = v\xi x_{01} + x_{02}, \tag{2.17}$$

and $\boldsymbol{v}_4$ to $\boldsymbol{v}_8$ equal to x_{11}, x_{12}, x_{13}, x_{22} and x_{33}. Then

$$\boldsymbol{\omega} = \alpha x_{00} + \beta[-2x_{01} + 2v\xi(x_{02} + x_{03}) + x_{23}] \tag{2.18}$$

and (2.3) gives

$$\mathrm{d}s^2 = \phi[2\,\mathrm{d}u\,\mathrm{d}v - 2v\xi\,\mathrm{d}u(\mathrm{d}\zeta + \mathrm{d}\bar{\zeta}) - \mathrm{d}\zeta\,\mathrm{d}\bar{\zeta}] + \psi\,\mathrm{d}u^2, \tag{2.19}$$

where ϕ, ψ, α and β are arbitrary scalars. Again an interpretation along the lines surrounding (2.14) can be made.

In this example and the previous one the bivector rank of the Riemann tensor is two and the metric is given algebraically up to two arbitrary functions. However, in the present example if the derivatives $R^\mu{}_{\nu\alpha\beta;\gamma}$ are known, then (1.7) and an obvious extension of the method yield

$$\boldsymbol{v}_9 = x_{00} + 2(U + 2v^2\xi^2)x_{01} \tag{2.20}$$

from which ω can be found and hence $\lambda g_{\mu\nu}$. Then

$$\mathrm{d}s^2 = \phi(2\,\mathrm{d}u\,\mathrm{d}v - 2U\mathrm{d}u^2 - |\mathrm{d}\zeta + 2v\xi\,\mathrm{d}u|^2), \tag{2.21}$$

where U is given by (2.16).

Example 4. One example of the Riemann tensor being spanned by two bivectors of the type (1.6) is given by

$$R^0{}_{001} = R^1{}_{110} = -2K_1(1 + K_1uv)^{-2}, \qquad R^2{}_{223} = R^3{}_{332} = -2K_2(1 + K_2\zeta\bar{\zeta})^{-2}, \tag{2.22}$$

where K_1 and K_2 are constants. These arise from the metric

$$\mathrm{d}s^2 = 2(1 + K_1uv)^{-2}\,\mathrm{d}u\,\mathrm{d}v - 2(1 + K_2\zeta\bar{\zeta})^{-2}\,\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} \tag{2.23}$$

(see McIntosh and Halford 1981, § 7 (vi)). Algebraically (2.22) and (1.1) give, for both

K_1 and K_2 non-zero,

$$ds^2 = \phi \, du \, dv + \psi \, d\zeta \, d\bar{\zeta}, \tag{2.24}$$

where ϕ and ψ are arbitrary scalars. On the other hand, if $K_1 = 0$ and $K_2 \neq 0$ in (2.22), then (1.1) gives

$$ds^2 = \alpha \, du^2 + \beta \, du \, dv + \gamma \, dv^2 + \psi \, d\zeta \, d\bar{\zeta}, \tag{2.25}$$

where α, β, γ and ψ are arbitrary scalars. In this particular case the Riemann tensor is spanned by one bivector, and four arbitrary scalars arise in the construction. This is the largest number of arbitrary scalars which can occur. A similar situation exists if $K_2 = 0$ and $K_1 \neq 0$.

Example 5. Newtonian gravity. On a four-dimensional manifold with coordinates $x^\mu = (t, x^i)$ $(i = 1, 2, 3)$, the equations which govern this theory are

$$d^2x^i/dt = -\partial\varphi/\partial x^i \equiv -\varphi_{,i}, \tag{2.26}$$

where φ is the potential function satisfying $\varphi_{,ii} = 4\pi\rho$. The non-zero components of the Riemann tensor are thus

$$R^i{}_{0j0} = -R^i{}_{00j} = \varphi_{,ij}. \tag{2.27}$$

It will now be shown by using the method outlined above that these components do not specify a non-degenerate four-dimensional metric. The non-zero expressions (2.2) are now twelve vectors in X:

$$\begin{aligned} &\varphi_{,11}x_{\mu 1} + \varphi_{,21}x_{\mu 2} + \varphi_{,31}x_{\mu 3}, \\ &\varphi_{,12}x_{\mu 1} + \varphi_{,22}x_{\mu 2} + \varphi_{,32}x_{\mu 3}, \\ &\varphi_{,13}x_{\mu 1} + \varphi_{,23}x_{\mu 2} + \varphi_{,33}x_{\mu 3}, \end{aligned} \tag{2.28}$$

for $\mu = 0, 1, 2, 3$. But of course these are not linearly independent. Provided that there are sufficiently many independent $\varphi_{,ij}$, the vectors in (2.28) with $\mu = 0$ span the x_{01}, x_{02}, x_{03} subspace. Thus $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ can be written as x_{01}, x_{02} and x_{03} respectively. Similarly the remaining vectors in (2.28) allow $\boldsymbol{v}_4$ to $\boldsymbol{v}_9$ to be written as x_{11}, x_{12}, x_{13}, x_{22}, x_{23} and x_{33} respectively. Notice that x_{00} does not appear in any of these expressions since there are no non-zero components $R^\mu{}_{\nu\alpha\beta}$ with $\mu = 0$. The requirement that the tenth vector $\boldsymbol{\omega}$ in X be orthogonal to these $\boldsymbol{v}_a$ would now give $\boldsymbol{\omega} = x_{00}$ and then (2.3) would yield $ds^2 = \psi \, dt^2$. In the cases where there are not sufficiently many $\varphi_{,ij}$ for this analysis to hold, this method would still give rise to degenerate metrics.

3. Comments

It is obvious from the first four examples that (1.1) yields $g_{\mu\nu}$, up to a conformal factor, from a given set of $R^\mu{}_{\nu\alpha\beta}$ almost always. The exceptions occur when the dimension of the bivector space of the curvature 2-form is three or less, and either there are one or two vectors $\boldsymbol{V}$ which satisfy (1.5) or else the curvature 2-form is spanned by two bivectors of the form (1.6). It is clear from the examples, and can obviously be shown in general, that:

(*a*) if there is one vector $\boldsymbol{V}$ which satisfies (1.5), then (1.1) determines the metric up to two arbitrary scalars (cf examples 2 and 3);

(*b*) if there are two vectors $\boldsymbol{V}$ which satisfy (1.5), then (1.1) determines the metric up to four arbitrary scalars (cf example 4, equation (2.25));

(*c*) if the curvature 2-form is spanned by two bivectors of the form (1.6), then (1.1) determines the metric up to two arbitrary scalars (cf example 4, equation (2.24));

(*d*) if none of the conditions (*a*)–(*c*) holds, then (1.1) determines the metric up to one arbitrary scalar, a conformal factor (cf example 1).

These aspects are further discussed by Hall and McIntosh (1981) and by McIntosh and van Leeuwen (1981).

It is also obvious that in the method of Ihrig (1975a) as modified in this paper the dimension of the vector space spanned by the $\boldsymbol{v}_a$ found from (2.2) is eight, six, eight and nine respectively in the cases (*a*)–(*d*) just mentioned. It cannot be less than six except in flat space, in which case it is zero. The number of arbitrary scalars in the coefficients of the most general vector $\boldsymbol{\omega}$ orthogonal to the $\boldsymbol{v}_a$ in each case is thus two, four, two and one respectively. This number is then the same as the number of arbitrary scalars in the metric.

Equation (1.3) with

$$x_{\mu\nu} = \xi_{(\mu;\nu)} \tag{3.1}$$

is a necessary condition that a metric admit a curvature collineation, i.e. a vector field $\boldsymbol{\xi}$ such that

$$\mathscr{L}_{\boldsymbol{\xi}} R^{\mu}{}_{\nu\alpha\beta} = 0, \tag{3.2}$$

where $\mathscr{L}$ denotes the Lie derivative.

For the majority of space–times none of the conditions (*a*)–(*c*) holds, (1.3) gives $x_{\mu\nu} = \phi g_{\mu\nu}$, and any curvature collineation is a conformal motion—see McIntosh (1981). The metrics in examples 2 and 3, however, always admit non-trivial curvature collineations; see Halford *et al* (1980) and the references given in that paper.

The method of obtaining the components $\lambda g_{\mu\nu}$ from the $R^{\mu}{}_{\nu\alpha\beta}$ as discussed in § 2 could in principle be carried out on a computer. A program to do this would probably follow Ihrig's (1975a) approach more closely. However, it is easy for the human eye to pick out a set of linearly independent vectors from the 96 given by (2.2) and rather wasteful to turn to a machine for this task. We hope that § 2, when followed through in practice, will convince the reader of this.

One can imagine *gedanken* experiments for determining the components of the Riemann tensor in some frame. Experiments along this line are discussed by Pirani (1965) and Szekeres (1965). The method of § 2 then gives in most cases the corresponding metric components up to a conformal factor.

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